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Whitehead's Integral Formula, Isolated Critical Points, and the Enhancement of the Milnor Number

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for Is Singer on his 85th birthday

Abstract: J. H. C. Whitehead gave an elegant integral formula for the Hopf invariant $\mathcal{H}(\mathbf{p})$ of a smooth map $\mathbf{p}$ from the 3-sphere to the 2-sphere. Given an open book structure $\mathbf{b}$ on the 3-sphere (or, essentially equivalently, an isolated critical point of a map F from $\mathbb{R}^4$ to $\mathbb{R}^2$), Whitehead's formula can be "integrated along the fibers" to express $\mathcal{H}(\mathbf{p})$ as the integral of a certain 1-form over S^1. In case $\mathbf{p}$ is geometrically related to $\mathbf{b}$ (or F)—for instance, if $\mathbf{p}$ is the map (one component of the fiberwise generalized Gauss map of F) whose Hopf invariant $\lambda(K)$ is the "enhancement of the Milnor number" of the fibered link $K \subset S^3$ associated to F (or $\mathbf{b}$), previously studied by the author and others—it might be hoped that this 1-form has geometric significance. This note makes that hope somewhat more concrete, in the form of several speculations and questions.

Keywords: Enhancement, fibered link, Hopf invariant, Milnor number, open book, Whitehead's integral formula

1. Whitehead's integral formula and isolated critical points

Let $\mathbf{p}\colon S^3 \to S^2$ be smooth. According to Whitehead [23] (see also [24, 2]), the Hopf invariant $\mathcal{H}(\mathbf{p})$ of $\mathbf{p}$ can be calculated as follows. Let Ω be the volume 2-form on S^2, normalized to give S^2 volume 1. Since Ω is closed, so is its pullback $\Omega_{\mathbf{p}} = \mathbf{p}^*(\Omega)$; since $H^2(S^3; \mathbb{R}) = \{0\}$, $\Omega_{\mathbf{p}}$ is in fact exact. Let $\eta_{\mathbf{p}}$ be a 1-form on S^3 with $d\eta_{\mathbf{p}} = \Omega_{\mathbf{p}}$. If g is any 0-form on S^3, then also $d(\eta_{\mathbf{p}} + dg) = \Omega_{\mathbf{p}}$, and since $H^1(S^3; \mathbb{R}) = \{0\}$, every solution η of $d\eta = \Omega_{\mathbf{p}}$ is of the form $\eta_{\mathbf{p}} + dg$. Thus, pointwise, the 3-form $\eta_{\mathbf{p}} \wedge \Omega_{\mathbf{p}}$ depends strongly on the choice of $\eta_{\mathbf{p}}$. Nonetheless, Whitehead showed that its integral is independent of this choice:

$$\int_{S^3} \eta_{\mathbf{p}} \wedge \Omega_{\mathbf{p}} = \mathcal{H}(\mathbf{p}). \tag{WIF}$$

It is clear that (WIF) remains valid, *mutatis mutandis*, if subjected to any of various modifications: the target of $\mathbf{p}$ could be $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ or $(\mathbb{R}^3 \setminus \{\mathbf{0}\}) \times \mathbb{R}$, the domain of $\mathbf{p}$ could be a punctured neighborhood of $\mathbf{0}$ in $\mathbb{R}^4$, and (in the latter case) the integration in (WIF) could be over any 3-cycle in the domain of $\mathbf{p}$ dual to the puncture at $\mathbf{0}$.

Let U be a neighborhood of $\mathbf{0}$ in $\mathbb{R}^4$, $F = (f, g)\colon U \to \mathbb{R}^2$ a map with $F(\mathbf{0}) = \mathbf{0}$ such that F is smooth on $U \setminus \{\mathbf{0}\}$ and F is continuous at $\mathbf{0}$. We will say that **$\mathbf{0}$ *is an isolated critical point of F*** provided that there exists $\delta > 0$ such that, if $0 < \|\mathbf{x}\| \leq \delta$, then the derivative $DF(\mathbf{x})$ has rank 2; and (following Kauffman & Neumann [8]) we will say that $\mathbf{0}$ is a *tame* isolated critical point of F if, further, for all sufficently small $\delta > 0$, (a) $F^{-1}(\mathbf{0})$ and δS^3 intersect transversely, and (b) there exists $\varepsilon(\delta) > 0$ such that, if $0 < \varepsilon \leq \varepsilon(\delta)$, then $F^{-1}(\varepsilon D^2) \cap \delta D^4$ is a topological 4-disk that is smooth except for corners along $F^{-1}(\varepsilon S^1) \cap \delta S^3$.

Examples. (a) If F is smooth at $\mathbf{0}$ and $\mathbf{0}$ is a regular point of F, then $\mathbf{0}$ is a tame isolated critical point of F. (b) If F is (the real-polynomial mapping underlying) a complex-polynomial mapping $(\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$ without repeated factors, then $\mathbf{0}$ is a tame isolated critical point of F. The differential topology of this case (and its higher-dimensional generalization), first studied by Milnor [14], continues to be a lively topic of investigation. (c) Let $F\colon (\mathbb{R}^4, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ be a real-polynomial mapping. For a generic such F, the real-algebraic set of critical points of F is 1-dimensional at each of its points, so $\mathbf{0}$ is an isolated critical point of F if and only if it is a regular point. Nonetheless, there exist many

real-polynomial mappings F for which $\mathbf{0}$ is a non-regular isolated critical point (necessarily tame, [8]). Among these mappings, those that underlie complex-polynomial mappings are neither the only, nor necessarily the most interesting, examples. (A wide variety of essentially non-complex examples are known; see [13, 16, 19, 18, 17, 1, 6].)

Proposition 1. *If $F = (f, g)\colon U \to \mathbb{R}^2$ has a tame isolated critical point at $\mathbf{0}$, and $\mathbf{p}\colon U \setminus \{\mathbf{0}\} \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$ is a smooth map, then for all sufficiently small $\delta > 0$,*

$$\mathcal{H}(\mathbf{p}) = \lim_{\varepsilon \to 0} \int_{\varepsilon S^1} \int_{F^{-1}(\varepsilon \mathbf{u}) \cap \delta D^4} \eta_{\mathbf{p}} \wedge \Omega_{\mathbf{p}} \tag{1}$$

Proof. By hypothesis (b) in the definition of tameness, the 3-manifold-with-corners $\partial(F^{-1}(\varepsilon D^2) \cap \delta D^4)$ (with its natural orientation) is a 3-cycle homologous to δS^3 in $U \setminus \{\mathbf{0}\}$, provided that δ and ε are sufficiently small. As a 3-cycle, $\partial(F^{-1}(\varepsilon D^2) \cap \delta D^4)$ is the sum of two 3-chains, $C_1(\delta, \varepsilon) = F^{-1}(\varepsilon S^1) \cap \delta D^4$ and $C_2(\delta, \varepsilon) = F^{-1}(\varepsilon D^2) \cap \delta S^3$; the latter, by hypothesis (a) in the definition of tameness, is a neighborhood of a smooth link $L = \delta S^3 \cap F^{-1}(\mathbf{0})$ in δS^3, and it is clear that for a fixed δ this neighborhood shrinks down to L as ε goes to 0. By the comments after (WIF),

$$\mathcal{H}(\mathbf{p}) = \int_{C_1(\delta, \varepsilon)} \eta_{\mathbf{p}} \wedge \Omega_{\mathbf{p}} + \int_{C_2(\delta, \varepsilon)} \eta_{\mathbf{p}} \wedge \Omega_{\mathbf{p}},$$

so

$$\mathcal{H}(\mathbf{p}) = \lim_{\varepsilon \to 0} \int_{C_1(\delta, \varepsilon)} \eta_{\mathbf{p}} \wedge \Omega_{\mathbf{p}};$$

this becomes (1) after integration along the fiber. $\qquad\square$

2. Whitehead's integral formula and open books

Tame isolated critical points are closely related to open-book structures. Here, an *open book* on S^3 is a smooth map $\mathbf{b}\colon S^3 \to \mathbb{R}^2$ such that $\mathbf{0}$ is a regular value of $\mathbf{b}$ and $\mathbf{b}/\|\mathbf{b}\|\colon S^3 \setminus \mathbf{b}^{-1}(\mathbf{0}) \to S^1$ is a fibration. Given an open book $\mathbf{b}$, it is easy to check that $\mathbf{0}$ is a tame isolated critical point of $co(\mathbf{b})\colon \mathbb{R}^4 \to \mathbb{R}^2$, where $co(\mathbf{b})(t\mathbf{x}) = t\mathbf{b}(\mathbf{x})$ for all $\mathbf{x} \in S^3$ and $t \geq 0$. Conversely [8], if $\mathbf{0}$ is a tame isolated critical point of $F\colon (U, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$, then there is an open book $\mathbf{b}_F$ (unique up

to an obvious equivalence relation) such that the isolated critical points at $\mathbf{0}$ of F and of $co(\mathbf{b}_F)$ are equivalent (again, in an obvious sense).

The *binding* of the open book $\mathbf{b}$ is $L_{\mathbf{b}} = \mathbf{b}^{-1}(\mathbf{0})$. The θth *page* of $\mathbf{b}$ is $S_{\mathbf{b}}(\theta) = \mathbf{b}^{-1}(\{t(\cos(\theta), \sin(\theta)) \mid t \geq 0\})$. It is immediate from the definitions that $L_{\mathbf{b}}$ is a smooth link (naturally oriented by the orientations of S^3 and S^2) and that each $S_{\mathbf{b}}(\theta)$ is a Seifert surface bounded by $L_{\mathbf{b}}$. In fact, the binding of an open book is a fibered link, and each page is a fiber surface for that link.

Proposition 2. *If* $\mathbf{b} \colon S^3 \to \mathbb{R}^2$ *is an open book, and* $\mathbf{p} \colon S^3 \to S^2$ *is a smooth map, then*

$$\mathcal{H}(\mathbf{p}) = \lim_{\varepsilon \to 0} \int_{S^1} \int_{S_{\mathbf{b}}(\theta) \,\setminus\, \mathbf{b}^{-1}(\varepsilon(\mathrm{Int}\, D^2))} \eta_{\mathbf{p}} \wedge \Omega_{\mathbf{p}} \tag{2}$$

Proof. This is entirely analogous to Proposition 1. □

3. The enhancement of the Milnor number

As in §1, let $F = (f, g) \colon (U, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ be smooth on $U \setminus \{\mathbf{0}\} \subset \mathbb{R}^4$ and continuous at $\mathbf{0}$. Easily, $\mathbf{0}$ is an isolated critical point of F if and only if there exists $\delta > 0$ such that, if $0 < \|\mathbf{x}\| \leq \delta$, then $(df \wedge dg)(\mathbf{x}) \neq 0$. In this case, the self-dual (resp., anti-self-dual) 2-form $df \wedge dg + \star(df \wedge dg)$ (resp., $df \wedge dg - \star(df \wedge dg)$), where $\star$ is the Hodge star operator with respect to the flat metric on $\mathbb{R}^4$, is non-degenerate; writing

$$df \wedge dg \pm \star(df \wedge dg) = A_{\pm}(dx \wedge dy + \pm du \wedge dv)$$
$$+ B_{\pm}(dx \wedge du \mp dy \wedge dv) + C_{\pm}(dx \wedge dv \pm dy \wedge du), \tag{3}$$

in terms of linear coordinates (x, y, u, v) on $\mathbb{R}^4$, non-degeneracy means that $(A_{\pm}, B_{\pm}, C_{\pm}) \colon \delta D^4 \setminus \{\mathbf{0}\} \to \mathbb{R}^3$ avoids $\mathbf{0}$, and so has a well-defined Hopf invariant. Define $\lambda(F) = \mathcal{H}(A_+, B_+, C_+)$, $\rho(F) = \mathcal{H}(A_-, B_-, C_-)$.

Lemma. $\lambda(F)$ *and* $\rho(F)$, *so defined, equal* $\lambda(F)$ *and* $\rho(F)$ *as defined in* [19].

Proof. Let G be the Grassmann manifold of oriented 2-planes in (oriented) $\mathbb{R}^4$. In [19], $(\lambda(F), \rho(F))$ is defined to be the homotopy class of the field of oriented 2-planes $\ker(DF)$, identified with a pair of integers via a standard choice of homeomorphism between G and $S^2 \times S^2$. The Plücker coordinates of $\ker(DF)$

are the 2×2 minor determinants of the 2×4 matrix that represents DF in coordinates (x, y, u, v), and each of the two factors of the map $\ker(DF)$ from a punctured neighborhood of $\mathbf{0}$ to $S^2 \times S^2$ is a certain sum or difference of two Plücker coordinates. For instance, the factor whose Hopf invariant is λ has coordinates $f_x g_y - f_y g_x + f_u g_v - f_v g_u$, $f_x g_u - f_y g_v - f_u g_x + f_v g_y$, and $f_x g_v + f_y g_u - f_u g_y - f_v g_x$, which by inspection are identical, respectively, to A_+, B_+, and C_+; and similarly for ρ. $\qquad\square$

In particular, if F is tame, then by [19]

$$\lambda(F) + \rho(F) = \mu(L_{\mathbf{b}_F}) \qquad (4)$$

is the *Milnor number* of the fibered link $L_{\mathbf{b}_F}$ associated to F, that is, the first Betti number of its fiber surface. Thus λ, ρ, and μ are, among them, only two independent invariants; following [15], we put ρ aside, calling $\lambda(F)$ the *enhancement of the fibered link $L_{\mathbf{b}_F}$* (or of its Milnor number), and writing $\lambda(L)$ for $\lambda(F)$.

The enhancement of a fibered link has been shown to have a number of interesting and useful properties, summarized in the following proposition. For proofs, see [20] and references cited therein.

Proposition 3. *Let S be a fiber surface in S^3, L the fibered link ∂S.*

(A) *The enhancement of L is not determined by its homological monodromy, and in particular it is not determined by the Seifert form of S.*

(B) *Let $Mir(L)$ denote the mirror image of L. Then $\lambda(Mir(L)) + \lambda(L) = \mu(L)$.*

(C) *Let S be a positive Hopf annulus (so that L is two fibers of a Hopf fibration $S^3 \to S^2$, oriented to have linking number $+1$). Then $\lambda(L) = 0$. Equivalently, if $F\colon \mathbb{C}^2 \to \mathbb{C}\colon (z, w) \mapsto zw$, then $\lambda(F) = 0$.*

(D) *As follows immediately from (B) and (C), if S is a negative Hopf annulus, then $\lambda(L) = 1$. Equivalently, if $F\colon \mathbb{C}^2 \to \mathbb{C}\colon (z, w) \mapsto z\overline{w}$, then $\lambda(F) = 1$.*

(E) *More generally than (C), if $F\colon (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$ is a complex polynomial without repeated factors, then $\lambda(F) = 0$.*

(F) *If S is a Murasugi sum of fiber surfaces S_1 and S_2, then $\lambda(L) = \lambda(\partial(S_1)) + \lambda(\partial(S_2))$.*

(G) *More generally than* (F), *if F unfolds into F_1 and F_2, in the sense of* [15], *then* $\lambda(F) = \lambda(F_1) + \lambda(F_2)$.

(H) *As follows immediately from* (C), (D), *and* (F), *if S is a Hopf-plumbed surface, then $\lambda(L)$ is the number of negative Hopf plumbands; in particular, in this case $0 \le \lambda(L) \le \mu(L)$.*

(I) *If a contact structure is associated to L by Giroux's construction* [3, 4], *then $\lambda(L)$ is the Hopf invariant of that contact structure; in particular, $\lambda(L)$ can be any integer.*

(J) *Generalizing* (H), *$\lambda(L)$ is the net number of negative Hopf plumbands in any stable Hopf plumbing of S (which exists, by* [3, 4]).

(K) *Let $e(\beta)$ denote the exponent sum of a braid $\beta \in B_n$. Given any β, construct a link L by adjoining to the closed braid $\widehat{\beta}$ the braid axis together with, for each component K of $\widehat{\beta}$, a nearby oppositely-oriented longitude $-K'$, where the linking number of K and K' is unrestricted. It is a beautiful observation of Hirasawa (see* [6]) *that then L is fibered and $\lambda(L)$ is equal to $n - e(\beta) + 1$, the Bennequin number of β.* □

4. Integral formulas for the enhancement

Applying the results in §1 to the definitions in §3, we can obtain integral formulas for the enhancement.

If one combines

$$\Omega_{(A_+, B_+, C_+)} = \frac{A_+ \, dB_+ \wedge dC_+ + B_+ \, dC_+ \wedge dA_+ + C_+ \, dA_+ \wedge dB_+}{(A_+^2 + B_+^2 + C_+^2)^{3/2}} \qquad (5)$$

with the formulas for A_+, B_+, and C_+ given in the Lemma, the easily confirmed formula

$$A_+^2 + B_+^2 + C_+^2 = (f_x^2 + f_y^2 + f_u^2 + f_v^2)(g_x^2 + g_y^2 + g_u^2 + g_v^2)$$
$$- (f_x g_x + f_y g_y + f_u g_u + f_v g_v)^2,$$

and (WIF), then—provided one has a method to produce a 1-form $\eta_{(A_+, B_+, C_+)}$ (there is no shortage of such methods, some more explicit than others)—one can derive a fearsome-looking formula for $\lambda(F)$ as the integral of what appears to me (perhaps wrongly) to be a pointwise-meaningless 3-form, which I forbear to display here.

Pressing on, one might take heart from the thought that the 2-form $\Omega_{(A_+,B_+,C_+)}$ seems to be intimately related to the fibers of F over which $\eta_{(A_+,B_+,C_+)} \wedge \Omega_{(A_+,B_+,C_+)}$ is integrated in formula (1). Although this formula might be manipulable into something palatable, I have not yet been able to establish anything along those lines.

A similarly regrettable situation, of high hopes dashed by intractable computations, prevails in the case of open books and formula (2).

Question. Are there, in fact, reasonable and meaningful integral formulas for λ in the general case? Such formulas do exist for μ. For instance, in the complex case, and in general dimensions, Kennedy [9] defines a "twisted tangent bundle" E_δ over the fiber $F^{-1}(\delta)$ ($\delta \neq 0$), with m^{th} Chern form $c_m(E_\delta)$, and (simplifying a formula of Griffiths [5]) proves that

$$1 - \mu(L) = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \int_{F^{-1}(\delta) \, \cap \, D_\varepsilon^{2m+2}} c_m(E_\delta).$$

See also [10, 11, 12]. In all these cases, the integrands are "pointwise-meaningful" insofar as they can be interpreted as various kinds of "curvatures". Does some such curvature interpretation exist for general isolated critical points?

5. Further questions

Even if no "meaningful" integral formula for λ can be found in the general case, perhaps there are interesting special cases where one exists. In particular, in light of items (H) and (I) in Proposition 3, it seems natural to ask for a (geometric or other) characterization of those fibered links L for which $0 \leq \lambda(L) \leq \mu(L)$.

Questions. (a) Can this class of fibered links be characterized using integral formulas? (b) Is this class of fibered links perhaps exactly those for which a mapping F can be found such that all fibers of F are minimal surfaces in $\mathbb{R}^4$?

Remark. Question (b) is emphatically not a conjecture. The main evidence for the affirmative answer is that, if F is a complex polynomial (or the mirror image of a complex polynomial), then all its fibers, being complex curves, are minimal surfaces. Some reason to think that the techniques of this paper might at least be relevant is that, according to Hoffman & Osserman [7], a necessary and sufficient condition for all the fibers of $F \colon (U, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ to be minimal is that

$\delta D^4 \backslash \{\mathbf{0}\} \to G : \mathbf{x} \mapsto \ker(df \wedge dg)$ be fiberwise (anti-)conformal, which puts strong restrictions on (A_+, B_+, C_+) and its anti-self-dual analogue (A_-, B_-, C_-). There is a considerable literature on topological restrictions of foliations by minimal surfaces, some of which might perhaps be adapted to the investigation of this question; in particular, "Rummler's Calculation" ([21], cited by Sullivan [22]) seems highly apposite.

In 1988, before Walter Neumann used the calculus of splice diagrams to work out an example (published in [15]) of a fibered link L with $\lambda(L) < 0$—and well before I noticed that, in fact, such links are plentiful among a family of examples already introduced in [19]—, it seemed reasonable (particularly in light of item (H) of Proposition 3) to wonder whether $\lambda(L)$ might be the dimension of some vectorspace naturally associated to L. At that time, Isadore Singer suggested to me that, even if it were never negative, it might nonetheless be better interpreted as the Euler characteristic of some complex (or perhaps the index of some operator), rather than as a dimension. Both suggestions seem even better to me now than they did then, but in 20 years I have made no progress on them. I do have, based on little more than wishful thinking, the notion that such a complex might be constructed by taking note of the complex structures naturally imposed on the fibers of F by their induced metrics; note the connection to the Hoffman–Osserman generalized Gauss map.

Question. Can some reader do better with Singer's suggestions?

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